Indian Statistical Institute, Bangalore
B.Math (Hons.) I Year, Second Semester
Backpaper Examination

Time: 3 hours | May 2012 | Instructor: C.R.E.Raja

Analysis II | Maximum marks: 50

Each question is worth 10 marks

1. Let (x_n) be a sequence in $\mathbb{R}^m$ and $x_n(i)$ be the i-th co-ordinate of x_n.

   (a) Prove that (x_n) converges if and only if $(x_n(i))$ converges for every i.

   (b) If (x_n) is bounded, then show that (x_n) has a convergent subsequence.

2. Let (X, d) be a metric space.

   (a) Prove that $|d(x, y) - d(a, b)| \leq d(x, a) + d(b, y)$ for any $a, b, x, y \in X$.

   (b) If (x_n) and (y_n) are Cauchy sequence in X, prove that $(d(x_n, y_n))$ converges in $\mathbb{R}$.

   (c) If (K_n) is a decreasing sequence of compact sets in X with $\text{diam}(K_n) \not\to 0$, prove that $\cap K_n$ has at least two points.

3. (a) If $f: [a, b] \to \mathbb{R}$ is a continuous function, prove that $f \in \mathcal{R}[a, b]$.

   (b) Let $f: [0, \infty) \to [0, \infty)$ be a decreasing function. Prove that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\sup_N \int_0^N f < \infty$.

4. (a) Suppose $a < c < b$ and $f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$. Show that $f \in \mathcal{R}[a, b]$.

   (b) Let $f \in \mathcal{R}[a, b]$ and define $g: [a, b] \to \mathbb{R}$ by $g(a) = 0$ and $g(x) = \int_a^x f$ for all $x \in (a, b]$. For any partition P of $[a, b]$, define $\Delta(g, P) = \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$ where $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ is the partition P. Show that $\int_a^b |f(t)|dt = \sup\{\Delta(g, P) \mid P \text{ is any partition of } [a, b]\}$.

5. (a) Let E be an open set in $\mathbb{R}^n$ and $f: E \to \mathbb{R}$ be a function that has local maximum at some $x \in E$. If $D_i f$ exists on E, prove that $D_i f(x) = 0$.

   (b) Let f and g be real-valued functions defined on $\mathbb{R}$ that have continuous second order derivatives. Define $F: \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x + g(y))$ for all $(x, y) \in \mathbb{R}^2$. Find a formula for the first and second order partial derivatives of F in terms of the derivatives of f and g and verify the relation $D_1 F D_{1,2} F = D_2 F D_{1,1} F$. Is F differentiable?